Full Input-Output Analysis for Economic and Environmental Impacts of An Arbitrary Sector

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Abstract

Leontief's Input-Output Analysis (LIOA), which regards final demands from several kinds of consumers as the right side of the equation (Y as in $X = (1 - A)^{-1} Y$), is extended to take demands from any economic sectors, as Y to study the impacts of these sectors. We call this approach, in which an arbitrary sector can be considered the external sector, Full Input-Output Analysis (FIOA), and we call those beyond LIOA as Targeted-sector Input-Output Analysis (TIOA, thus LIOA + TIOA = FIOA). When FIOA is applied to a combined system of economic sectors and environmental sectors, a new formula different from the one commonly used by researchers, to measure environmental impacts is derived. Both TIOA and the new formula are applied to toy-mode data, and the results are shown to be either more accurate or more informative than those from conventional formulae. Furthermore, formulae for multiregion FIOA are also developed and their application is illustrated on toy-mode data, in which the economic and environmental impacts of an arbitrary economic sector in each region or in combined regions are calculated.

Keywords: Input-output analysis, Environmental impacts

Highlights

1. A Full Input-Output Analysis (FIOA), in which an arbitrary sector can be taken as the external sector, is proposed.
2. When FIOA is applied to a combined system of economic sectors and environmental sector, a new formula in a fully decomposition form for measuring environmental impacts is derived.

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3. Both FIOA and the new formula for measuring environmental impacts are applied to toy-mode data, and the results are shown to be more accurate or more informative than those from conventional formulae.
4. Formulae for multiregion FIOA are also developed and their application is illustrated on toy-mode data.

Contents

1. Introduction

Leontief's input-output analysis (LIOA) (Leontief, W, 1970, Miller & Blair, 2009) has been applied to quantify the economic and environmental impacts of input-output flow between economic sectors within the same country and between countries embedded in global trade (Turner et al., 2007, Wiedmann et al., 2007). In most cases, the original form of LIOA is used,

$$X \triangleq LY = (1 - A)^{-1} Y, \tag{1}$$

where Y is a vector of the amount (in physical or monetary units) of each product demanded by the final demanders, which are often considered to be consumers and governments. When applied to environmental impact analysis (Leontief, W, 1970, Hubacek et al., 2017, Acquaye et al., 2017, Ayres & Kneese, 1969, Turner et al., 2007, Wiedmann et al., 2007, 2015), a vector $f^{e_j}_i$, which represents the amount of waste released into or the amount of resources extracted from environment e_j per unit of product i, is required and the net environmental impacts to e_j is calculated by

$$X^{e_j} \triangleq f^{e_j} LY = \sum_{ik} f^{e_j}_i L^i_k Y^k. \tag{2}$$

We call this an added-on approach of measuring environmental impacts.

Note that the idea of input-output analysis is to consider both direct and indirect input-output relations among all sectors, captured in the following expression,

$$X = (1 - A)^{-1} Y = Y + AY + A^2 Y + A^3 Y + \cdots . \tag{3}$$

It means that, to produce Y, the economy must first produce Y, then the direct input to produce such a Y thus AY, and the direct input to produce such an AY thus A^2Y, and so on. Thus, in total, $X = (1 - A)^{-1} Y$. Then $f^{e_j}Y, f^{e_j}AY, f^{e_j}A^2Y, \cdots$ stand for the waste or resources linked to the production of each corresponding term.

Therefore, while Eq. (2), following the added-on approach, is fully reasonable, a natural idea is to treat environments as sectors of a combined input-output table of both economic and environmental sectors and apply the above idea of including both direct and indirect connections to redevelop a set of formulae for input-output analysis. We call this the Full Input-Output Analysis (FIOA) approach. In this work, we will take this full approach and check whether the derived formulae are different from Eq. (2); furthermore, if they are different, then we test the significance of this difference using toy-mode data.

2. Methods

Let us illustrate the general formula using an economy with two producer sectors (c_1, considered for example as "agriculture" and c_2, considered for example as "industry"), one final demander (fd, considered for example as consumers) and two environmental sectors (e_1, considered for example as "clean water" and e_2, considered for example as "wasted water"), as shown in Table 1. x^i_j represents the materials flow from sector i to sector j. For example, $x^{c_1}_{c_2}$ means the amount of product C_1 used to produce product C_2 and $x^{c_1}_{e_1}$ represents the amount of product C_1 directly going into environment e_1 as waste, while $x^{e_1}_{c_1}$ refers to the amount of resource E_1, such as clean water, extracted from the environment to produce product C_1. Note that the byproduct waste from the economic sector, say c_1, released into the environment e_2 is denoted as an input from environment, such as $x^{e_2}_{c_1}$. In terms of this notation, the manufacturing of product C_1 can be described as

$$C_1 + C_2 + E_1 + E_2 \,(\text{"-"sign}) + Fd \rightarrow C_1. \tag{4}$$

For the current analysis, we might want to take

$$x^{e_i}_{e_j} = 0, \tag{5}$$

since these values are about the environmental physical/chemical processes themselves, instead of the relationship between economy and environments, although in principle they might not be zero: say for example, wasted water (e_2) might become clean water (e_1) via some environmental processes,

$$E_2 \rightarrow E_1 \,(\text{ignored}). \tag{6}$$

Of course, if one wants to include environmental processes with economic process and if there are such empirical data, then these terms can be added back to the table.

$x^{fd}_{c_j}$ is the value added from the labor force to economic sector c_j. $x^{fd}_{e_j}$ represents the amount of labor directly needed for the environment to produce clean/wasted water. Note that water factories are already taken into account by economic sector c_2; thus the environment itself does not require labor to generate clean water. Thus, $x^{fd}_{e_j} = 0$. We further neglect x^{fd}_{fd}, which represents the labor needed to reproduce labor. It might be nonzero in principle, but it should be very small when compared to $x^{c_j}_{fd}$. That is, we ignore Fd in the following process of regenerating the workforce,

$$C_1 + C_2 + E_1 + E_2 \,(\text{"-"sign}) + Fd \,(\text{ignore this term}) \rightarrow Fd. \tag{7}$$

$x_{e_k}^{c_j}$ might also be considered to be zero since it represents the direct release of product C_j into environment e_k. In practice, this process might occur, for example, unemployment and pouring milk directly into rivers. However, we believe that those terms are much smaller than the other terms in the input-output table, thus

$$C_1 \to E_1 \,(\text{ignored})\,. \tag{8}$$

After all of these simplifications, we arrive at the input-output table shown in Table 1.

Table 1: An illustration of the sectors and their input-output relationship. c1 and c2 are economic sectors, such as industry and agriculture and fd is the final demander. e1 and e2 are environmental sectors.

from \ to	c_1	c_2	fd	e_1	e_2
c_1	$x_{c_1}^{c_1}$	$x_{c_2}^{c_1}$	$x_{fd}^{c_1}$	0	0
c_2	$x_{c_1}^{c_2}$	$x_{c_2}^{c_2}$	$x_{fd}^{c_2}$	0	0
fd	$x_{c_1}^{fd}$	$x_{c_2}^{fd}$	0	0	0
e_1	$x_{c_1}^{e_1}$	$x_{c_2}^{e_1}$	$x_{fd}^{e_1}$	0	0
e_2 ("-"sign)	$x_{c_1}^{e_2}$	$x_{c_2}^{e_2}$	$x_{fd}^{e_2}$	0	0

In principle, all of the terms in the input-output table can be nonzero. However, neither this possibility changes the formulae that we are going to develop and nor it invalidates the idea of FIOA.

More economic sectors and environmental sectors, as well as more types of final demanders, can be added to this table. Furthermore, when global trade is also considered, this table can be expanded, for example c_j become c_j^r, meaning sector j in country (region) r. Further examples will be used to illustrate this global trade situation.

2.1. Environmental impact: New terms arise from waste and resources needed to reproduce labor

Let us now develop the formula. As in LIOA, we define the total output of each sector j as,

$$X^i = \sum_j x_j^i, \tag{9}$$

and define direct input-output coefficients as

$$B^i_j = \frac{x^i_j}{X^j}, \tag{10}$$

meaning that, to produce each product of sector j, an amount of input from sector i is required. In all analyses, we consider these coefficients to be constant for the interested period of time.

First, let us apply the idea of FIOA to this input-output table regarding fd as the external sector, similar to the standard LIOA. Then for those $i \neq fd$, we have

$$X^i = \sum_{j\neq fd} x^i_j + x^i_{fd} = \sum_{j\neq fd} \frac{x^i_j}{X^j} X^j + x^i_{fd} \Rightarrow X = BX + Y \Rightarrow X = L^{(-fd)}Y, \tag{11}$$

where

$$L^{(-fd)} = \left(1 - B^{(-fd)}\right)^{-1}, \tag{12}$$

$$Y^i = x^i_{fd}. \tag{13}$$

This equation is a general one: once we have L, given arbitrary Y, we can always calculate the required X and this X can of course include X^{e_j}. In principle, we already have the necessary formula, which looks exactly like the formula in standard LIOA, but different in its meaning since X includes both X^{c_i} and X^{e_j}.

In deriving the above formula, two key ideas are important: First, total output of a sector (X^i) is defined as the summation of all of the output from this sector to all of the sectors in the system, and second, by converting this definition into a linear equation assuming that the coefficient matrix (B^{-fd}) is a constant, we know both the direct and indirect effects of final demands ($Y^i = x^i_{fd}$). Recall Eq. (3) to see why both the direct and indirect effects are included in Eq. (11). We call this two key ideas the input-output principle.

However, we do not want to stop at the general Eq. (11). Considering the zeros in Table 1 and looking at X^{e_j} in particular, these zeros might further simplify the above L^{-fd} and lead to a simpler formula for X^{e_j}. Writing down the equations one by one, we have

$$X^{c_1} = x^{c_1}_{c_1} + x^{c_1}_{c_2} + x^{c_1}_{fd} = B^{c_1}_{c_1}X^{c_1} + B^{c_1}_{c_2}X^{c_2} + Y^{c_1}, \tag{14a}$$

$$X^{c_2} = x^{c_2}_{c_1} + x^{c_2}_{c_2} + x^{c_2}_{fd} = B^{c_2}_{c_1}X^{c_1} + B^{c_2}_{c_2}X^{c_2} + Y^{c_2}, \tag{14b}$$

$$X^{fd} = x^{fd}_{c_1} + x^{fd}_{c_2} = B^{fd}_{c_1}X^{c_1} + B^{fd}_{c_2}X^{c_2}, \tag{14c}$$

$$X^{e_1} = x^{e_1}_{c_1} + x^{e_1}_{c_2} + x^{e_1}_{fd} = B^{e_1}_{c_1}X^{c_1} + B^{e_1}_{c_2}X^{c_2} + B^{e_1}_{fd}X^{fd}, \tag{14d}$$

$$X^{e_2} = x^{e_2}_{c_1} + x^{e_2}_{c_2} + x^{e_2}_{fd} = B^{e_2}_{c_1}X^{c_1} + B^{e_2}_{c_2}X^{c_2} + B^{e_2}_{fd}X^{fd}. \tag{14e}$$

and we find that the first two equations form a closed linear equation already, and the remaining three equations all rely on X^{c_1} and X^{c_2}, which can be solved from the first two equations. Thus,

$$X^{c_j} = \sum_{c_k} L_{c_k}^{(-e-fd),c_j} Y^{c_k}, \tag{15}$$

where $L^{(-e-fd)} = \left(1 - B^{(-e-fd)}\right)^{-1}$ is the original Leontief inverse, where only economic sectors and not environmental sectors are considered. Here, the sign $(-e - fd)$ means to eliminate the environmental sectors (e) and the sector of final demanders (fd) from the whole matrix B,

$$B^{(-e-fd)} = \begin{bmatrix} B_{c_1}^{c_1} & B_{c_2}^{c_1} \\ B_{c_1}^{c_2} & B_{c_2}^{c_2} \end{bmatrix}. \tag{16}$$

Once we have X^{c_j}, the remaining three questions are readily solved, meaning that we do not need to inverse the matrix anymore: rather, we have

$$X^{fd} = \sum_{c_j} B_{c_j}^{fd} X^{c_j} = \sum_{c_j,c_k} B_{c_j}^{fd} L_{c_k}^{(-e-fd),c_j} Y^{c_k}, \tag{17}$$

and in turn from Eq. (14d) and Eq. (14e), we obtain

$$X^{e_i} = \sum_{c_j,c_k} B_{c_j}^{e_i} L_{c_k}^{(-e-fd),c_j} Y^{c_k} + B_{fd}^{e_i} \sum_{c_j,c_k} B_{c_j}^{fd} L_{c_k}^{(-e-fd),c_j} Y^{c_k}. \tag{18}$$

This formula is one of the central formulae of our FOIA. Compared with Eq. (2), which is rephrased in the current notation as,

$$X^{e_i} = \sum_{c_j,c_k} B_{c_j}^{e_i} L_{c_k}^{(-e-fd),c_j} Y^{c_k}, \tag{19}$$

Eq. (18) has one more term,

$$B_{fd}^{e_i} \sum_{c_j,c_k} B_{c_j}^{fd} L_{c_k}^{(-e-fd),c_j} Y^{c_k} = B_{fd}^{e_j} X^{fd}, \tag{20}$$

which represents the waste and resources related to labor X^{fd} required to produce the product Y^{c_k}. Of course, if $B_{fd}^{e_i}$, meaning the wastes and resources required to reproduce one unit of labor, is very small, then this term can be ignored. Therefore, how large the difference is between Eq. (18) and Eq. (2)/Eq. (19) must be checked by comparing results from the two formulae in practice.

We want to emphasize that FIOA applies exactly the idea of original LIOA, i.e. the input-output principle, to a combined system of economic and environmental sectors. Instead of using Eq. (2) from the added-on approach to measure the environmental impact, what we do is simply rederive the formula for the combined system, and then Eq. (18) appears naturally.

Recently, Hubacek et al. (Hubacek et al., 2017) added one more term to Eq. (2),

$$X^{e_j} = \sum_{ik} f_i^{e_j} L_k^i Y^k + hh_{\text{dir}}, \tag{21}$$

which in terms of our notation becomes

$$X^{e_i} = \sum_{c_j, c_k} B_{c_j}^{e_i} L_{c_k}^{(-e-fd),c_j} Y^{c_k} + B_{fd}^{e_i} X^{fd}, \tag{22}$$

where X^{fd} is the amount of workforce (value added) from all households. Both Eq. (18) and Eq. (22) take into account the idea of considering both the environmental impacts embedded in the manufacturing of products and also in value-added sectors. The difference between the two lies in that Eq. (18) is in a decomposition form, thus, it is applicable for any given Y^{c_k}, while the second term of Eq. (22) is only at the aggregated level. Compared against Eq. (22), in our Eq. (18), X^{fd} is further decomposed into value added required by each economic sector.

Defining a matrix $\bar{\mathcal{L}}^{-e-fd_1}$ as follows simplifies and unifies Eq. (15), Eq. (17) and Eq. (18)

$$\bar{\mathcal{L}}^{-e-fd} = \left[\begin{array}{c|c} \multirow{2}{*}{L^{-e-fd}} & 0 \\ & 0 \\ \hline B_{-fd}^{fd} L^{-e-fd} & 1 \end{array}\right], \tag{23}$$

where

$$B_{-fd}^{fd} \triangleq B_{c_1,c_2}^{fd} = \left[B_{c_1}^{fd}, B_{c_2}^{fd}\right], \tag{24}$$

then

$$\begin{bmatrix} X^{c_1} \\ X^{c_2} \\ X^{fd} \end{bmatrix} = \bar{\mathcal{L}}^{-e-fd} \begin{bmatrix} Y^{c_1} \\ Y^{c_2} \\ Y^{fd} \end{bmatrix} \tag{25}$$

which we simply denote as

$$X^{\text{econ}} = \bar{\mathcal{L}}^{-e-fd} Y^{\text{econ}}. \tag{26}$$

With this short hand matrix notation, environmental impacts can be denoted as

$$X^{e} = B^{e}_{\text{econ}} X^{\text{econ}} = \left(B^{e}_{\text{econ}} \bar{\mathcal{L}}^{-e-fd}\right) Y^{\text{econ}}, \tag{27}$$

where

$$B^{e}_{\text{econ}} = \begin{bmatrix} B^{e_1}_{c_1} & B^{e_1}_{c_2} & B^{e_1}_{fd} \\ B^{e_2}_{c_1} & B^{e_2}_{c_2} & B^{e_2}_{fd} \end{bmatrix}. \tag{28}$$

2.2. Targeted-sector input-output analysis for pure economic systems

Second, let us apply the idea of LIOA to the same input-output table, regarding an arbitrary sector l as the external sector. For simplicity, for the moment, let us first forget about the environmental sectors, that is, to consider a pure economic system, as Table 2. We return to the combined economics-environmental system later.

Table 2: An illustration of the sectors and their input-output relationship. c_1 and c_2 are economic sectors, such as agriculture and industry. fd is the sector of final demanders.

from \ to	c_1	c_2	fd
c_1	$x^{c_1}_{c_1}$	$x^{c_1}_{c_2}$	$x^{c_1}_{fd}$
c_2	$x^{c_2}_{c_1}$	$x^{c_2}_{c_2}$	$x^{c_2}_{fd}$
fd	$x^{fd}_{c_1}$	$x^{fd}_{c_2}$	0

Consider a sector l as the external sector. Then, for those $i \neq l$, we have

$$X^{i} = \sum_{j \neq l} x^{i}_{j} + x^{i}_{l} = \sum_{j \neq l} \frac{x^{i}_{j}}{X^{j}} X^{j} + x^{i}_{l} \Rightarrow X = BX + Y \Rightarrow X = L^{(-l)} Y, \tag{29}$$

where

$$L^{(-l)} = \left(1 - B^{(-l)}\right)^{-1}, \tag{30}$$

$$Y^{i} = x^{i}_{l}. \tag{31}$$

What can be revealed from this equation? Let us use $l = c_1$ as an example and ask what the meaning is of $L_j^{(-c_1)}$. Denote the vector that only has element 1 at the jth place and everywhere else 0,

$$\delta Y_{c_1}^j = \left[0, \cdots, 0, 1_j, 0, \ldots, 0\right]^T, \tag{32}$$

meaning one unit of input from sector j is needed by sector c_1. Then, to satisfy this need, the whole economy will need to produce

$$L_j^{(-c_1)} = L^{(-c_1)} \delta Y_{c_1}^j. \tag{33}$$

Therefore, $L_j^{(-c_1),i}$ indicates how much of product i the whole economy must produce to satisfy the needs of one unit of input from sector j required by sector c_1.

In this notation, in fact, the original LIOA becomes the special case of taking $l = fd$,

$$L = L^{(-fd)}. \tag{34}$$

Accordingly, the meaning of $L_j^{(-fd),i}$ is how much of product i the whole economy must produce to satisfy the needs of one unit of input from j required by fd. By taking $l = fd$, the column vector $L_j^{(-fd)}$ ($L^{(-fd)}[:, j]$) represents the integrated direct and indirect effect on the whole economy due to consumers' demands for each product j. Its element $L_j^{(-fd),i}$($L^{(-fd)}[i, j]$) can be seen as a response from/pressure on sector i due to a one-unit increased in demands for product j from final demanders. $L_j^{(-fd),i}$ can also be regarded as a decomposition of X^i due to various final demands, as in $X^i = \sum_j L_j^{(-fd),i} x_{fd}^j$. Response and decomposition are the two typical ways of looking at the inverse matrix. Similarly, that is also the intuitive meaning of $L_j^{(-c_1),i}$: as a response to a one-unit increase in input of product j as required by sector c_1, or as a decomposition of all X^i due to various inputs required by sector c_1.

$L^{(-c_1)}$ and $L^{(-fd)}$ are exactly parallel, and each is a special case of $L^{(-l)}$. $L^{(-c_j)}$ is called the Targeted-sector Input-Output Analysis (TIOA), in which an arbitrary sector other than fd can be treated as the external sector. Combining $L^{(-c_1)}$ and $L^{(-fd)}$, we use whichever sector is the sector of interest as the external sector, as in $L^{(-l)}$. Thus, FIOA = LIOA + TIOA, meaning that $L^{(-l)}$ refers to either $L^{(-c_1)}$ or $L^{(-fd)}$.

With this TIOA, we are able to answer the following questions, to provide each unit of product j required by the production of the sector l, which could be

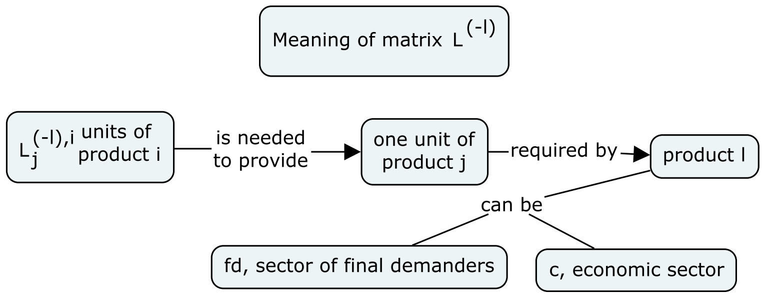

Figure 1: Meaning of each element of matrix $L^{(-l)}$, as a response or as a decomposition.

an economic sector c_l or a final demanders sector fd (or a c_l^r or a fd^r in a region r), how many units of all products are needed, as illustrated in Fig. 1.

By extending $L^{(-fd)}$ to $L^{(-l)}$, not only can the economic impacts of the needs from final demanders be investigated, but also the economic impacts of the needs from other economics sectors can be discussed. When global trade is considered, we can discuss the economic impacts of sector c_2^r (c_2 in region r) using $L^{(-c_2^r)}$. Questions like this can be what will happen to the agricultural sector in the USA if **the Chinese industrial sector** for some reason requires less mail service. Previous studies often have focused on the economic impacts on sector c_1 in region I from final demanders in region II or worldwide (Miller & Blair, 2009), such as what will happen to the agricultural sector in the USA if **Chinese consumers** for some reason require less mail service or wear less jewelry.

We also use toy-model data to provide some examples of what can we know from performing this extended analysis in a multiregion setting.

2.3. Targeted-sector input-output analysis for combined economic and environmental systems

Let us consider c_1 in Table 1 as the external sector and rederive the formula corresponding to Eq. (18).

$$X^{c_1} = x_{c_1}^{c_1} + x_{c_2}^{c_1} + x_{fd}^{c_1} = Y^{c_1} + B_{c_2}^{c_1} X^{c_2} + B_{fd}^{c_1} X^{fd}, \tag{35a}$$

$$X^{c_2} = x_{c_1}^{c_2} + x_{c_2}^{c_2} + x_{fd}^{c_2} = Y^{c_2} + B_{c_2}^{c_2} X^{c_2} + B_{fd}^{c_2} X^{fd}, \tag{35b}$$

$$X^{fd} = x_{c_1}^{fd} + x_{c_2}^{fd} = Y^{fd} + B_{c_2}^{fd} X^{c_2}, \tag{35c}$$

$$X^{e_1} = x_{c_1}^{e_1} + x_{c_2}^{e_1} + x_{fd}^{e_1} = B_{c_1}^{e_1} X^{c_1} + B_{c_2}^{e_1} X^{c_2} + B_{fd}^{e_1} X^{fd}, \tag{35d}$$

$$X^{e_2} = x_{c_1}^{e_2} + x_{c_2}^{e_2} + x_{fd}^{e_2} = B_{c_1}^{e_2} X^{c_1} + B_{c_2}^{e_2} X^{c_2} + B_{fd}^{e_2} X^{fd}. \tag{35e}$$

We can see that the above equation can again be decomposed into two parts and that the middle two equations are coupled together; from these observation we can find $\left[X^{c_2}, X^{fd}\right]^T$ from $\left[Y^{c_2}, Y^{fd}\right]^T = \left[x_{c_1}^{c_2}, x_{c_1}^{fd}\right]^T$ and the remaining equations are then readily solved,

$$\begin{bmatrix} X^{c_2} \\ X^{fd} \end{bmatrix} = L^{(-e-c_1)} \begin{bmatrix} Y^{c_2} \\ Y^{fd} \end{bmatrix} \tag{36}$$

where

$$L^{(-e-c_1)} = \left(1 - B^{(-e-c_1)}\right)^{-1} \tag{37}$$

and

$$B^{(-e-c_1)} = \begin{bmatrix} B_{c_2}^{c_2} & B_{fd}^{c_2} \\ B_{c_2}^{fd} & 0 \end{bmatrix}. \tag{38}$$

With these solutions, we can arrive at

$$X^{c_1} = Y^{c_1} + \left[B_{c_2}^{c_1}, B_{fd}^{c_1}\right] L^{(-e-c_1)} \begin{bmatrix} Y^{c_2} \\ Y^{fd} \end{bmatrix} \tag{39}$$

and finally the environmental impact due to $\left[Y^{c_2}, Y^{fd}\right]^T = \left[x_{c_1}^{c_2}, x_{c_1}^{fd}\right]^T$ is

$$X^{e_j} = B_{c_1}^{e_j} Y^{c_1} + B_{c_1}^{e_j} \left[B_{c_2}^{c_1}, B_{fd}^{c_1}\right] L^{(-e-c_1)} \begin{bmatrix} Y^{c_2} \\ Y^{fd} \end{bmatrix} + \left[B_{c_2}^{e_j}, B_{fd}^{e_j}\right] L^{(-e-c_1)} \begin{bmatrix} Y^{c_2} \\ Y^{fd} \end{bmatrix}. \tag{40}$$

Defining a complete input-output matrix $\bar{\mathcal{L}}^{-e-c_1}$ as following simplifies Eq. (36), Eq. (39) and Eq. (40)

$$\bar{\mathcal{L}}^{-e-c_1} = \left[\begin{array}{c|c} 1 & B_{-c1}^{c_1} L^{-e-c_1} \\ \hline 0 & \\ 0 & L^{-e-c_1} \end{array}\right], \tag{41}$$

where

$$B_{-c_1}^{c_1} \triangleq B_{c_2,fd}^{c_1} = \left[B_{c_2}^{c_1}, B_{fd}^{c_1}\right]. \tag{42}$$

Then,

$$\begin{bmatrix} X^{c_1} \\ X^{c_2} \\ X^{fd} \end{bmatrix} = \bar{\mathcal{L}}^{-e-c_1} \begin{bmatrix} Y^{c_1} \\ Y^{c_2} \\ Y^{fd} \end{bmatrix}, \tag{43}$$

which we simply denote as

$$X^{\text{econ}} = \bar{\mathcal{L}}^{-e-c_1} Y^{\text{econ}}. \tag{44}$$

With this short-hand matrix notation, environmental impacts can be calculated using again Eq. (27).

This environmental impact is on environmental sector e_j, and it is due to the input from sector c_2 or fd required by sector c_1. This question is beyond the usual studies of environmental impact and Eq. (40) is another central formula of this work. Note that both key formulae Eq. (18) and Eq. (40) come from the general input-output principle that captured in Eq. (29). The only difference is that fd and c_1 is regarded as the external sector in respectively Eq. (18) and Eq. (40). The reason that Eq. (18) has one less term is because $x_{fd}^{fd} = 0$ while $x_{c_1}^{c_1} \neq 0$. In practice, often $x_{c_1}^{c_1}$ is small compared with $x_{c_2}^{c_1}, x_{fd}^{x_1}$, thus, we may sometimes neglect this Y^{c_1} term.

In principle, any sectors, including economic sectors, final demanders, and also even environmental sector, can be treated as the external sector of input-output analysis. However, in this work, we do not consider the environmental sectors as the external sectors. Why? There is no need to consider economic impact or environmental impact of an input flow to the environmental sectors. In fact, all the input flow from other sectors to the environmental sectors, according to our assumption in this work, is considered to be zero $x_{e_j}^{i} = 0$. This assumption means that, even if we regard e_j as the external sector, the corresponding external vector will always be zero ($Y = 0$). Thus, even if we calculate the corresponding Leontief inverse L^{-e_j}, $L^{-e_j} Y = 0$. $L^{-e_j} = \frac{1}{1-B^{-e^j}}$ will also become ill defined since matrix B^{-e^j} is still an input-output matrix of the economics system thus it has an eigenvalue of 1 ($B^{-e^j} X = X$) and when a matrix B has an eigenvalue of 1 then $\frac{1}{1-B}$ is ill defined.

The above formula of targeted input-output analysis for combined systems can also be performed on input-output tables with global trade by adding regional information. We again illustrate via examples what kind of additional information can be extracted by applying this full input-output analysis with global trade in combined economics-environment systems with an arbitrary sector as the external sector.

2.4. *On the matrix notation*

In matrix notation, the two sets of formulae, i.e. Eq. (13), Eq. (23), Eq. (26) and Eq. (27) for the case of fd as the external sector, and Eq. (37), Eq. (41), Eq. (44) and Eq. (27) for the case of c_1 as the external sector, are exactly parallel. These notations are even more abstract than the formulae in element forms, such as Eq. (18) and Eq. (29). Thus, there is an even greater cognitive burden in reading and using these matrix notations. However, we maintain these formulae because the two sets of formulae share the same structures, which does tell us that the two methods are from exactly the same principle, and the only difference is that different sectors are used as the external sectors.

Furthermore, the matrix notation also has practical value. Starting with the full matrix of direct input-output matrix B, it is easy and straightforward to obtain B^{-e} and B^{e}_{econ}. Once the external sector is chosen, e.g., sector l, then it is again straightforward to obtain B^{-e-l} and B^{l}_{-l}, as in

$$B = \left[\begin{array}{cc|c} B^{l}_{l} & B^{l}_{-l} & \multirow{2}{*}{$B^{\text{econ}}_{e} = 0$} \\ B^{-e-l}_{l} & B^{-e-l} & \\ \hline \multicolumn{2}{c|}{B^{e}_{\text{econ}}} & B^{e}_{e} = 0 \end{array}\right]. \tag{45}$$

Then, with the chosen value of l, LIOA simply converts the sub input-output matrix for the economic system only

$$B^{-e} = B^{\text{econ}}_{\text{econ}} = \left[\begin{array}{c|c} B^{l}_{l} & B^{l}_{-l} \\ \hline B^{-e-l}_{l} = Y^{-e-l} & B^{-e-l} \end{array}\right] \tag{46}$$

into a decomposition form of pure economic impacts of an input-output flow from any sector j, including l itself, to sector l, which is described by the jth column $(\bar{\mathcal{L}}^{-e-l}[:, j])$ of

$$\bar{\mathcal{L}}^{-e-l} = \left[\begin{array}{c|c} 1 & B^{l}_{-l}L^{-e-l} \\ \hline 0 & \multirow{2}{*}{L^{-e-l}} \\ 0 & \end{array}\right]. \tag{47}$$

From here, a decomposition form of the environmental impacts of an input-output flow from any sector j, including l itself, to sector l is described by the jth column $(\mathcal{L}^{e-l}[:, j])$ of

$$\mathcal{L}^{e-l} = B^{e}_{\text{econ}}\bar{\mathcal{L}}^{-e-l}. \tag{48}$$

These two matrices can then be used to find economic and environmental impact via $X^{\text{econ}} = \bar{\mathcal{L}}^{-e-l}Y^{\text{econ}}$ and $X^{e} = \mathcal{L}^{e-l}Y^{\text{econ}}$. Therefore, all of the information is encoded in these two matrices.

This holds for arbitrary l. In real calculations, l might not be the first sector, so before and after calculation, some rearrangement of the orders of the sectors must be done. In fact, we believe that this unified formula in abstract matrix form is much easier to apply than it is to find properly the formula for various purposes of each application.

3. Results on toy-mode data

In this section, using toy-model data and choosing various sectors as the external sector (the sector l as in L^{-l}), we illustrate the application and merit of the input-output principle, which is captured in Eq. (29), or equivelently in Eq. (47) and Eq. (48).

3.1. Toy-mode data

Our toy-model system has three economic sectors, respectively c_1 (agriculture), c_2 (industry) and fd (consumers), and two environmental sectors, i.e. e_1 (clean water resource supply) and e_2 (wasted water). The input-output table reads line by line as follows: c_1 produces rice and supplies 1 units (tons) of rice to c_1 (perhaps as seeds, for instance), 20 units to c_2 and finally 10 units to fd. Thus, the total output of c_1 is 31 units. Similarly, c_2 produces 80 units (bolts) of cloth in total, $10, 20, 50$ which goes into c_1, c_2 and fd, and fd supplies 600 units (hours) of labor into c_1 (200 units) and c_2 (400 units).

In producing rice, c_1 requires 10 units (tons) of clean water from the environment (denoted as e_1) and releases 9 units of then-waste water back into the environment (e_2). Similarly, c_2 gets 20 units of clean water from and releases 15 units of waste water back into the environments, and fd uses 5 units of clean water from e_1 and dumps 4 units of waste water to e_2. The numbers of resources and wastes are chosen to be slightly different to consider that some water goes into products, such as in rice and cloth, and some into human bodies. Differences between these numbers do not really matter.

We want to mention that we intentionally set $x_j^j \ll X^j$, meaning that the input-output flow from sector j to itself is much less than the total output of the sector. Again, as of the applicability of the methods, this choice of numbers does not matter. However, we think that if indeed every sector produces only one product, like chemicals in chemical reactions, then such self-generating terms should be much smaller than the total output to other sectors.

Empirical data on input-output table and also emission vectors can be found from The Organisation for Economic Co-operation and Development (OECD),

Table 3: A contrived system with an economic system, which itself includes two production sectors, one sector of final demands (consumers), and two environmental sectors. The first environmental sector supplies clean water to the economic sectors, which in turn releases waste water into the second environmental sector.

from \ to	agri	indu	fd	row sum
c_1, agri (rice, ton)	1	20	10	31
c_2, indu (fabric, bolt)	10	20	50	80
fd, consumers (labor, hour)	200	400	0	600
e_1, water supply (water, ton)	10	20	5	35
e_2,water waste("-"sign, water, ton)	9	15	4	28

World Input-Output Database (WIOD), Eora global supply chain database and other data courses that used in for example Mi et al. (2018), Acquaye et al. (2017), Hubacek et al. (2017) *etc*. An illustration applying the methods to these empirical data will be the topic of a future investigation.

3.2. Pure economic single-region input-output analysis

Let us first consider the economic sectors only, which concerns the first three rows and columns of Table 3. For this system, the full input-output coefficient matrix B, which is the starting point of all calculation, is

$$B = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} & \frac{10}{600} \\ \frac{10}{31} & \frac{100}{80} & \frac{50}{600} \\ \frac{200}{31} & \frac{400}{80} & 0 \end{bmatrix}. \tag{49}$$

We apply the standard LIOA, Eq. (11), and TIOA, Eq. (29), to discuss pure economic impacts of the sector of final demands and of the economic sector c_1.

3.2.1. Final demands as the external sector

Let us first illustrate the application of the standard LIOA, Eq. (11), which is a special case of Eq. (29). Note that this standard LIOA can be found in any standard LIOA textbooks (see, for example, Leontief, W (1970), Miller & Blair (2009)).

When fd is regarded as the external sector, the direct input-output coefficient matrix becomes,

$$B^{(-fd)} = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} \\ \frac{10}{31} & \frac{100}{80} \end{bmatrix}, \tag{50}$$

which removes the sector fd, hence the notation B^{-fd} or more explicitly both $B^{fd}_{c_1,c_2}$ and B^{c_1,c_2}_{fd}, from the full input-output coefficient matrix B.

From B^{-fd}, we calculate from Eq. (11) the Leontief inverse,

$$L^{(-fd)} = \frac{1}{1 - B^{(-fd)}} = \begin{bmatrix} \frac{93}{80} & \frac{31}{80} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}, \tag{51}$$

Thus, when

$$Y = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \tag{52}$$

we get

$$X = L^{(-fd)}Y = \begin{bmatrix} 31 \\ 80 \end{bmatrix}, \tag{53}$$

which agrees exactly with the total output from sector c_1 and c_2.

In a decomposition form, when respectively

$$\delta Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{54}$$

we get correspondingly

$$\delta X = L^{(-fd)}\delta Y = \begin{bmatrix} \frac{93}{80} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{31}{80} \\ \frac{3}{2} \end{bmatrix}. \tag{55}$$

We can check this is a proper decomposition by examining,

$$10 \begin{bmatrix} \frac{93}{80} \\ \frac{1}{2} \end{bmatrix} + 50 \begin{bmatrix} \frac{31}{80} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 31 \\ 80 \end{bmatrix}. \tag{56}$$

This decomposition reads as follows: if for some reason one (1) more unit of product $C_1(C_2)$ is needed by final demanders, then the whole economy will have to produce $\frac{93}{80} > 1$ ($\frac{31}{80}$) unit of product C_1 and $\frac{1}{2}$ ($\frac{3}{2} > 1$) units of product C_2.

From this process, we can clearly see the power of the decomposition form of LIOA and the results due to the idea of incorporating both direct and indirect input-output flows such that, when the final demanders require only one unit of product $C_1(C_2)$, the economy produces usually more than that one unit of this product.

Following this line of thinking, we next want to ask the following question: what is the economic impact if, for some reason a sector c_1 needs one fewer (or more) unit of product C_2 or one fewer (or more) hour of labor from sector fd? For this purpose, we need TIOA, captured in Eq. (29), and treat sector c_1 as the external sector.

3.2.2. An economic sector as the external sector

When sector c_1 is regarded as the external sector, we need to remove the row and column corresponding to c_1, thus,

$$B^{(-c_1)} = \begin{bmatrix} \frac{10}{31} & \frac{100}{80} \\ \frac{200}{31} & \frac{400}{80} \end{bmatrix}. \tag{57}$$

From B^{-c_1}, via Eq. (29), we calculate the Leontief inverse,

$$L^{(-c_1)} = \frac{1}{1 - B^{(-c_1)}} = \begin{bmatrix} 3 & \frac{1}{4} \\ 15 & \frac{9}{4} \end{bmatrix}. \tag{58}$$

Thus, when

$$Y = \begin{bmatrix} 10 \\ 200 \end{bmatrix}, \tag{59}$$

we get

$$X = L^{(-c_1)} Y = \begin{bmatrix} 80 \\ 600 \end{bmatrix}, \tag{60}$$

which agrees exactly with the total output from sector c_1 and fd.

In a decomposition form, when respectively

$$\delta Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{61}$$

we obtain correspondingly

$$\delta X = L^{(-c_1)} \delta Y = \begin{bmatrix} 3 \\ 15 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{9}{4} \end{bmatrix}. \tag{62}$$

We can confirm this is a proper decomposition by examining,

$$10\begin{bmatrix}3\\15\end{bmatrix}+200\begin{bmatrix}\frac{1}{4}\\\frac{9}{4}\end{bmatrix}=\begin{bmatrix}80\\600\end{bmatrix}. \tag{63}$$

This decomposition reads as follows: if for some reason one more unit of product $C_2(fd)$ is needed by c_1, then the whole economy will have to produce $3 > 1$ ($\frac{1}{4}$) units of product C_2 and 15 ($\frac{9}{4} > 1$) hours of labor from fd.

We see that, by applying TIOA, Eq. (29), we know the response of the whole economy to a single unit of product or labor required by economic sector c_1. Similarly, we can use sector c_2 as the external sector to study the response of the whole economy to a single unit of product or labor required by sector c_2. This goes beyond the traditional LIOA, in which we always use fd as the external sector; thus, in tranditional LIOA only the response of the whole economy to a single unit of product consumed by final demanders can be investigated.

When environmental sectors are combined together with economic sectors to form a combined system, the same input-output principle and formula can be used to study impacts to the whole combined system from any sector. This process will be illustrated in the next subsection.

3.3. Single-region environmental impacts

To consider environmental impacts, let us use the combined system in Table 3.

The direct input-output coefficient matrix of this input-output table is

$$B=\begin{bmatrix}\frac{1}{31} & \frac{20}{80} & \frac{10}{600}\\ \frac{10}{31} & \frac{20}{80} & \frac{50}{600}\\ \frac{200}{31} & \frac{400}{80} & 0\\ \frac{10}{31} & \frac{20}{80} & \frac{5}{600}\\ \frac{9}{31} & \frac{15}{80} & \frac{4}{600}\end{bmatrix}. \tag{64}$$

We apply our Eq. (18) for environmental impacts, and the results are compared to those from Eq. (19) and Eq. (22).

3.3.1. Due to final demands

Let us first apply Eq. (18), which relies on $L^{(-e-fd)}=\left(1-B^{(-e-fd)}\right)^{-1}$, where

$$B^{(-e-fd)}=\begin{bmatrix}\frac{1}{31} & \frac{20}{80}\\ \frac{10}{31} & \frac{20}{80}\end{bmatrix}. \tag{65}$$

Thus,

$$L^{(-e-fd)} = \begin{bmatrix} \frac{93}{80} & \frac{31}{80} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}. \tag{66}$$

Note that

$$Y^{(-e-fd)} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \tag{67}$$

and the emission matrix is

$$B_c^e = \begin{bmatrix} \frac{10}{31} & \frac{20}{80} \\ \frac{9}{31} & \frac{15}{80} \end{bmatrix}, B_{fd}^e = \begin{bmatrix} \frac{5}{600} \\ \frac{4}{600} \end{bmatrix} \tag{68}$$

. Note that all of the above $B^{(-e-fd)}, L^{(-e-fd)}$ and $Y^{(-e-fd)}$ have been found in the previous subsection.

In reality, data on total X^{e_j} (here 35 and 28 respectively) might not be provided to researchers, although in our contrived system, they are available. What researchers do know is this emission matrix B^e. Now, we calculate the whole amount of waste released to and total resources extracted from environment from $L^{(-e-fd)}$, $Y^{(-e-fd)}$ and B^e via Eq. (18), Eq. (19) or Eq. (22), and we compare the results to the total amount available in the contrived system.

From Eq. (19), we have

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B_c^e L^{(-e-fd)} Y^{(-e-fd)} = \begin{bmatrix} 30 \\ 24 \end{bmatrix} \neq \begin{bmatrix} 35 \\ 28 \end{bmatrix}, \tag{69}$$

from which, we can see that there is a difference between the calculated and known amounts of resources and wastes.

Next, let us try Eq. (22). For that we need

$$X^{fd} = 600. \tag{70}$$

Then we get

$$\begin{bmatrix} X^{e_{ws}} \\ X^{e_{ww}} \end{bmatrix} = B_c^e L^{(-e-fd)} Y^{(-e-fd)} + B_{fd}^e X^{fd} = \begin{bmatrix} 35 \\ 28 \end{bmatrix}. \tag{71}$$

These calculated values agree exactly with the known values.

It is time now to put Eq. (18) to a test. To do so, we need the value-added coefficient matrix (amount of value added from final demanders to other economic sectors),

$$B^{fd}_{c_1,c_2} = \begin{bmatrix} \frac{200}{31} & \frac{400}{80} \end{bmatrix} \triangleq B^{fd}. \tag{72}$$

Then, we arrive at

$$X^{e_i} = B^e_c L^{(-e-fd)} Y^{(-e-fd)} + B^e_{fd} B^{fd} L^{(-e-fd)} Y^{(-e-fd)} = \begin{bmatrix} 35 \\ 28 \end{bmatrix}. \tag{73}$$

Again, it agrees with the known values exactly.

Overall, we see that, both Eq. (22) proposed in (Hubacek et al., 2017) and our Eq. (18) lead to exact results, while the conventional formula Eq. (19) does not. The difference is mainly due to the direct emissions from consumers. These direct emission might often be smaller than the emissions from the agriculture and industry sector, but it is still better to have a systematic approach to take them into consideration. Furthermore, as we mentioned earlier, our Eq. (18) is in a decomposition form, while Eq. (22) is in an aggregated form. Let us now use examples to illustrate what additional information can be revealed using the decomposition form.

Taking respectively

$$\delta Y^{-e-fd} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{74}$$

we want to know how much emissions are due to the consumption of a single unit of product C_1/C_2. Eq. (22) is not applicable to this situation. Eq. (18) tells us,

$$\delta X^{e_i} = \begin{bmatrix} \frac{7}{12} \\ \frac{239}{480} \end{bmatrix}, \begin{bmatrix} \frac{7}{12} \\ \frac{221}{480} \end{bmatrix} \tag{75}$$

while Eq. (19) gives us,

$$\delta X^{e_i} = \begin{bmatrix} \frac{1}{2} \\ \frac{69}{160} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{63}{160} \end{bmatrix}. \tag{76}$$

This decomposition from Eq. (18) (not that from Eq. (19)) can be recombined to show that it is indeed the correct decomposition,

$$10 \begin{bmatrix} \frac{7}{12} \\ \frac{239}{480} \end{bmatrix} + 50 \begin{bmatrix} \frac{7}{12} \\ \frac{221}{480} \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \end{bmatrix}, \tag{77a}$$

$$10 \begin{bmatrix} \frac{1}{2} \\ \frac{69}{160} \end{bmatrix} + 50 \begin{bmatrix} \frac{1}{2} \\ \frac{63}{160} \end{bmatrix} = \begin{bmatrix} 30 \\ 24 \end{bmatrix}. \tag{77b}$$

Those two are different since Eq. (19) is missing a term, which is properly taken into account by Eq. (18). As of the difference between Eq. (22) and Eq. (18), Eq. (22) simply can not reveal information in a decomposition form.

The decomposition-form result from Eq. (18) indicates that, in order to supply one ton of rice to final demanders, the whole environment must provide $\frac{7}{12}$ tons of clean water and absorb $\frac{239}{480}$ tons of waste water. Similarly, in order to supply one bolt of fabric to final demanders, the whole environment needs to provide $\frac{7}{12}$ tons of clean water and absorb $\frac{221}{480}$ tons of waste water.

3.3.2. Due to other economic sectors

Let us now choose c_1 the agriculture sector as the external sector and apply Eq. (40). We need $L^{(-e-c_1)} = \left(1 - B^{(-e-c_1)}\right)^{-1}$, where

$$B^{(-e-c_1)} = \begin{bmatrix} \frac{100}{80} & \frac{50}{600} \\ \frac{400}{80} & 0 \end{bmatrix}. \tag{78}$$

Thus,

$$L^{(-e-c_1)} = \begin{bmatrix} 3 & \frac{1}{4} \\ 15 & \frac{9}{4} \end{bmatrix}. \tag{79}$$

Note that

$$Y^{(-e-c_1)} = \begin{bmatrix} 10 \\ 200 \end{bmatrix}, \tag{80}$$

and the emissions matrix is

$$B^{e}_{-c_1} = B^{e}_{c2,fd} = \begin{bmatrix} \frac{20}{80} & \frac{5}{600} \\ \frac{15}{80} & \frac{4}{600} \end{bmatrix}, B^{e}_{c_1} = \begin{bmatrix} \frac{10}{31} \\ \frac{9}{31} \end{bmatrix}. \tag{81}$$

Notations are defined according to the convention that A^{-c} or A_{-c} means removing terms about sector c from the upper or the lower indices of matrix A.

We also need the amount of product C_1 from the c_1 sector to other economic sectors (c_2, fd),

$$B^{c_1}_{-c_1} = B^{c_1}_{c2,fd} = \begin{bmatrix} \frac{20}{80} & \frac{10}{600} \end{bmatrix} \tag{82}$$

and input from sector c_1 it itself,

$$Y^{c_1} = x^{c_1}_{c_1} = \begin{bmatrix} 1 \end{bmatrix}. \tag{83}$$

Then, from Eq. (40) we arrive at

$$X^{e_i} = B^{e}_{-c_1} L^{(-e-c_1)} Y^{(-e-c_1)} + B^{e}_{c_1} B^{c_1}_{-c_1} L^{(-e-c_1)} Y^{(-e-c_1)} + B^{e}_{c_1} Y^{c_1} = \begin{bmatrix} 35 \\ 28 \end{bmatrix}. \tag{84}$$

Again, it agrees with the known values exactly. If the last term, which comes from $Y^{c_1} = x^{c_1}_{c_1}$, is neglected, we obtain

$$X^{e_i} = B^{e}_{-c_1} L^{(-e-c_1)} Y^{(-e-c_1)} + B^{e}_{c_1} B^{c_1}_{-c_1} L^{(-e-c_1)} Y^{(-e-c_1)} = \begin{bmatrix} 34.7 \\ 27.7 \end{bmatrix}, \tag{85}$$

which is, although different, still very close to the exact result. This is due to the choice of the data in the toy model such that $x^j_j \ll X^j$.

Eq. (84) can also be regarded as an equation in a decomposition form, in which the environmental impacts of a single unit of $Y^{c_1} = x^{c_1}_{c_1}$, $Y^{c_2} = x^{c_2}_{c_1}$ or $Y^{fd} = x^{fd}_{c_1}$, correspond to

$$\begin{bmatrix} \delta Y^{c_1} \\ \delta Y^{c_2} \\ \delta Y^{fd} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{86}$$

as follows,

$$\delta X^{e_i} = \begin{bmatrix} \frac{10}{31} \\ \frac{9}{31} \end{bmatrix}, \begin{bmatrix} \frac{297}{248} \\ \frac{2363}{2480} \end{bmatrix}, \begin{bmatrix} \frac{563}{4960} \\ \frac{4509}{49600} \end{bmatrix}. \tag{87}$$

This decomposition can be recombined to show that it is indeed the correct decomposition,

$$1 \begin{bmatrix} \frac{10}{31} \\ \frac{9}{31} \end{bmatrix} + 10 \begin{bmatrix} \frac{297}{248} \\ \frac{2363}{2480} \end{bmatrix} + 200 \begin{bmatrix} \frac{563}{4960} \\ \frac{4509}{49600} \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \end{bmatrix} \tag{88}$$

This outcomes means that, if the agriculture sector for some reason needs one bolt fewer of cloth from the industry, then the whole environment burden will be $\frac{297}{248}$ tons and $\frac{2363}{2480}$ tons less of clean and waste water. Similarly, if the agriculture sector for some reason needs one hour fewer labor from the final demanders, then the whole environment burden will be $\frac{563}{4960}$ ton and $\frac{4509}{49600}$ ton less of clean and waste water.

3.4. Multiregion environmental impacts

Next, we illustrate the application of FIOA to the economic and environmental impacts of a multiregion economic and environmental system. For that, we use a contrived system as in Table 4 with two regions (I, II), each of which has one economic sector (c^I and c^{II}), one final demander (fd^I and fd^{II}) and one environmental sector (e^I and e^{II}).

The table is compiled to be as close as possible to Table 3 simply by splitting the fd sector in Table 3 into two fd sectors (fd^I and fd^{II}) in Table 4. We perform the splitting by setting $x_{c^I}^{fd^I} = 200$ and $x_{c^{II}}^{fd^{II}} = 400$, instead of $x_{c^I}^{fd^I} = 100 = x_{c^{II}}^{fd^I}$ and $x_{c^{II}}^{fd^{II}} = 200 = x_{c^I}^{fd^{II}}$, since we want the workforce of each region to mainly work in its own region. For the same reason, we set $x_{c^{II}}^{e^I} = 0$ such that rivers in region I, e^I, do not directly supply clean water to region II. Then, the $x_{c_2}^{e_1} = 20$ in Table 3 is added to $x_{c^{II}}^{e^{II}} = 15 + 20 = 35$. However, if there is no input-output flow between the two regions, they will be totally separated. Therefore, we keep all of the other import-export flows, such as $x_{fd^{II}}^{c^I} = 5$ and $x_{c^{II}}^{c^I} = 20$ and so on.

Regarding $x_{fd^{II}}^{c^I} = 5$, in real statistical data, this term might be counted as an input-output flow from c^I to c^{II} and then from c^{II} to fd^{II}, indicating that product C^I in region I is first imported by sector c^{II} into region II and then consumers in region II use these products, which are now counted as part of product C^{II} (see, for example Eq(3.5) on page 78 of Miller & Blair (2009)). In the above contrived system, we count this part of product C^I as a flow from c^I to fd^{II}. Whether or not such separation, which distinguishes product C^I being imported into region II to be the part that indeed becomes raw materials to produce C^{II} or to be the part that directly goes into the sector of final demanders fd^{II}, can be done in reality, is not of our concern in this work. Furthermore, regardless of the way in which counting is used to construct such an input-output table, it does not invalidate our analysis. Of course, the interpretation of the calculated results depends on which counting method is used.

By the way, if the alternative counting is used, then the following modification should be made to Table 4: $x_{fd^{II}}^{c^I} = 5-5 = 0$, $x_{c^{II}}^{c^I} = 20+5 = 25$, $x_{fd^{II}}^{c^{II}} = 25+5 = 30$ and $x_{fd^I}^{c^{II}} = 25 - 25 = 0$, $x_{c^I}^{c^{II}} = 10 + 25 = 35$, $x_{fd^I}^{c^I} = 5 + 25 = 30$.

Table 4: A contrived system with two regions, each of which has one economic sector, one final demander and one environmental sector. $x^{C^I}_{C^{II}}$, $x^{C^I}_{fd^{II}}$, $x^{C^{II}}_{C^I}$ and $x^{C^{II}}_{fd^I}$ are the import-export between the two regions. Note that the two environmental sectors in each region can be merged when necessary. We intentionally made the waste released from C^I to e^{II} nonzero ($x^{e^{II}}_{C^I} = 9$), while $x^{e^I}_{C^{II}} = 0$. The values in this table are chosen to be a replicate of Table 3 except that the fd sector is divided into equally into two sectors fd^I and fd^{II}.

from \ to	c^I	fd^I	e^I	c^{II}	fd^{II}	e^{II}	row sum
c^I	1	5	0	20	5	0	31
fd^I	200	0	0	0	0	0	200
e^I	10	5	0	0	0	0	15
c^{II}	10	25	0	20	25	0	80
fd^{II}	0	0	0	400	0	0	400
e^{II}	9	0	0	35	4	0	48

Thus, the full input-output coefficient matrix is

$$B = \begin{bmatrix} \frac{1}{31} & \frac{5}{200} & 0 & \frac{20}{80} & \frac{5}{400} & 0 \\ \frac{200}{31} & 0 & 0 & 0 & 0 & 0 \\ \frac{10}{31} & \frac{5}{200} & 0 & 0 & 0 & 0 \\ \frac{10}{31} & \frac{25}{200} & 0 & \frac{20}{80} & \frac{25}{400} & 0 \\ 0 & 0 & 0 & \frac{400}{80} & 0 & 0 \\ \frac{9}{31} & 0 & 0 & \frac{35}{80} & \frac{4}{400} & 0 \end{bmatrix}. \tag{89}$$

We now apply FIOA to this multiregion economic and environmental system. Note that the key difference between this multiregion analysis and the single-region analysis in the previous subsections is to treat a sector in one of the region as the external sector. Of course, the external section can also be the combined sector of the final demanders of both regions $fd = fd^I + fd^{II}$ as usually done in multiregion input-output analysis (Miller & Blair, 2009).

3.4.1. Due to final demands

First, we illustrate the conventional perspective in multiregion input-output analysis. That is to regard all the sectors of final demanders from both regions as a whole. Subsequently, we will address the final demanders in each individual region and determine what additional information can be revealed.

Grouping together the two sectors of final demanders fd^I and fd^{II}

Let us first taking fd, which is the combination of fd^I and fd^{II} as the external sector, also regard environment e to be the combination of e^I and e^{II}, we have

$$B^{(-e-fd)} = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} \\ \frac{10}{31} & \frac{20}{80} \end{bmatrix}. \tag{90}$$

Thus,

$$L^{(-e-fd)} = \begin{bmatrix} \frac{93}{80} & \frac{31}{80} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}. \tag{91}$$

Eq. (18) also requires

$$Y^{(-e-fd)} = \begin{bmatrix} 10 \\ 50 \end{bmatrix} \tag{92}$$

and

$$B_c^{fd} = \begin{bmatrix} \frac{200}{31} & 0 \\ 0 & \frac{400}{80}. \end{bmatrix} \tag{93}$$

The emission matrices are

$$B_c^e = \begin{bmatrix} \frac{10}{31} & 0 \\ \frac{9}{31} & \frac{35}{80} \end{bmatrix}, B_{fd}^e = \begin{bmatrix} \frac{5}{200} & 0 \\ 0 & \frac{4}{400} \end{bmatrix}. \tag{94}$$

Each of these numbers in these matrices reads as sector c^I release waste of $\frac{5}{200}$ units into sector e^I for production of per unit of product C_1, final demanders fd^{II} release waste $\frac{4}{400}$ units into sector e^{II} for the reproduction of one hour of labor, and so on.

Then, via Eq. (18), we obtain

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B_c^e L^{(-e-fd)} Y^{(-e-fd)} + B_{fd}^e B_c^{fd} L^{(-e-fd)} Y^{(-e-fd)} = \begin{bmatrix} 15 \\ 48 \end{bmatrix}. \tag{95}$$

This outcome again agrees exactly with the known values of waste released to the environment. However, if only the first term is included, then

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B_c^e L^{(-e-fd)} Y^{(-e-fd)} = \begin{bmatrix} 10 \\ 44 \end{bmatrix} \neq \begin{bmatrix} 15 \\ 48 \end{bmatrix}, \tag{96}$$

which is quite different from the above exact result.

Per one unit product C^I or C^{II} consumed by fd, we have

$$\delta Y^{(-e-fd)} = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}. \tag{97}$$

Then, the corresponding waste to the environment is respectively

$$\begin{bmatrix}\delta X^{e^I}\\ \delta X^{e^{II}}\end{bmatrix} = \begin{bmatrix}\frac{9}{16}\\ \frac{93}{160}\end{bmatrix}, \begin{bmatrix}\frac{3}{16}\\ \frac{27}{32}\end{bmatrix}. \tag{98}$$

We can see that c^{II} makes a much greater impact on environment e^{II} than that of c^{II} on e^I ($\delta X^{e^{II}}_{c^{II}} = \frac{27}{32} > \delta X^{e^I}_{c^{II}} = \frac{9}{16}$) and than that of c^I on e^{II} ($\delta X^{e^{II}}_{c^{II}} = \frac{27}{32} > \delta X^{e^{II}}_{c^I} = \frac{93}{160}$). This is expected since $x^{e^{II}}_{c^{II}} = 35$ is much larger than $x^{e^I}_{c^{II}} = 0$ and also than $x^{e^{II}}_{c^I} = 9$. What is worth noting is that

$$\delta X^{e_I}_{c^{II}} > 0, \text{while } x^{e_I}_{c^{II}} = 0, \tag{99}$$

which means that while sector c^{II} does not direct release any waste into e^I, there is an environment impact on e^I from per product C^{II} consumed by fd. Why does this happen? It is because sector C^{II} must import from C^I, which releases waste directly into e^I, and also because the production of C^I requires a labor force from fd^I, which also releases waste into e^I.

This is exactly the beauty of input-output analysis: both direct and indirect effects are considered in a single formula, and it is in a decomposition form so that the impact of any unit of an input-output flow can be calculated.

Impact from final demanders fd^I only

Now, we consider fd^I as the external sector and we have

$$B^{\left(-e-fd^I\right)} = \begin{bmatrix}\frac{1}{31} & \frac{20}{80} & \frac{5}{400}\\ \frac{10}{31} & \frac{20}{80} & \frac{25}{400}\\ 0 & \frac{400}{80} & 0\end{bmatrix}. \tag{100}$$

Thus,

$$L^{\left(-e-fd^I\right)} = \begin{bmatrix}\frac{217}{160} & \frac{31}{32} & \frac{31}{400}\\ 1 & 3 & \frac{1}{5}\\ 5 & 15 & 2\end{bmatrix}. \tag{101}$$

Eq. (18) also requires

$$Y^{\left(-e-fd^{I}\right)} = \begin{bmatrix} 5 \\ 25 \\ 0 \end{bmatrix} \tag{102}$$

and

$$B^{fd^{I}}_{c^{I},c^{II},fd^{II}} \triangleq B^{fd^{I}}_{-fd^{I}} = \begin{bmatrix} \frac{200}{31} & 0 & 0 \end{bmatrix} \tag{103}$$

The emission matrices are

$$B^{e}_{c^{I},c^{II},fd^{II}} \triangleq B^{e}_{-fd^{I}} = \begin{bmatrix} \frac{10}{31} & 0 & 0 \\ \frac{9}{31} & \frac{35}{80} & \frac{4}{400} \end{bmatrix}, B^{e}_{fd^{I}} = \begin{bmatrix} \frac{5}{200} \\ 0 \end{bmatrix}. \tag{104}$$

Then, via Eq. (18), we obtain

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B^{e}_{-fd^{I}} L^{\left(-e-fd^{I}\right)} Y^{\left(-e-fd^{I}\right)} + B^{e}_{fd^{I}} B^{fd^{I}}_{-fd^{I}} L^{\left(-e-fd^{I}\right)} Y^{\left(-e-fd^{I}\right)} = \begin{bmatrix} 15 \\ 48 \end{bmatrix}. \tag{105}$$

This outcome again agrees exactly with the known values of waste released into the environment. However, if only the first term is taken into account, then

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B^{e}_{-fd^{I}} L^{\left(-e-fd^{I}\right)} Y^{\left(-e-fd^{I}\right)} = \begin{bmatrix} 10 \\ 48 \end{bmatrix} \neq \begin{bmatrix} 15 \\ 48 \end{bmatrix}, \tag{106}$$

which is quite far from the above exact results.

Per one unit product C^{I}, C^{II}, or fd^{II} consumed by fd^{I}, we have

$$\delta Y^{\left(-e-fd^{I}\right)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{107}$$

Then, the corresponding waste to the environment is respectively

$$\begin{bmatrix} \delta X^{e^{I}} \\ \delta X^{e^{II}} \end{bmatrix} = \begin{bmatrix} \frac{21}{32} \\ \frac{141}{160} \end{bmatrix}, \begin{bmatrix} \frac{15}{32} \\ \frac{279}{160} \end{bmatrix}, \begin{bmatrix} \frac{3}{80} \\ \frac{13}{100} \end{bmatrix}. \tag{108}$$

These decomposition values are very informative. For example, as a response to an input-output flow from C^{II} to fd^{I}, which is an import-export term between the two regions, waste released to environments e^{I} and e^{II} is, respectively, $\frac{15}{32}$ and $\frac{279}{160}$. These amounts are the environmental burden due to exporting one more unit of product of C^{II} to fd in region I. Only from this decomposition, such a response to arbitrary input-output flow can be investigated.

In this subsection, we have illustrated that, when there are multiple regions, the final demanders of each region or of the combined region can be used as the external sector to study the environmental impacts of an arbitrary unit of flow from other sectors to those various choices of final demanders. Note that all of the formulae and the ideas, i.e. the spirit of input-output analysis, remain the same as in previous subsections, and they are applied to the full system. The only difference is the choice of the external sectors.

Following the same line of thinking, instead of focusing on the environmental impacts of input to various final demanders, we might want to know what the environmental impacts are if one more hour of labor from region II is used to produce product C^I in region I (one unit in $x_{C^I}^{fd^{II}}$). For this purpose, we must treat C^I as the external sector, which is the task of the next subsection.

3.4.2. Due to other economic sectors

Besides fds, other economic sectors can also be treated as the external sector to study responses to the needs of these sectors. As an example, let us use c^I as the external sector,

$$B^{(-e-c^I)} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{25}{200} & \frac{20}{80} & \frac{25}{400} \\ 0 & \frac{400}{80} & 0 \end{bmatrix}. \tag{109}$$

Thus,

$$L^{(-e-c^I)} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{7} & \frac{16}{7} & \frac{1}{7} \\ \frac{10}{7} & \frac{80}{7} & \frac{12}{7} \end{bmatrix}. \tag{110}$$

Eq. (40) also requires

$$Y^{(-e-c^I)} = \begin{bmatrix} 200 \\ 10 \\ 0 \end{bmatrix}, \tag{111}$$

$$B_{fd^I,c^{II},fd^{II}}^{c^I} \triangleq B_{-c^I}^{c^I} = \begin{bmatrix} \frac{5}{200} & \frac{20}{80} & \frac{5}{400} \end{bmatrix} \tag{112}$$

and

$$Y^{c^I} = x_{c^I}^{c^I} = \begin{bmatrix} 1 \end{bmatrix}. \tag{113}$$

The emission matrices are

$$B^e_{fd^I, c^{II}, fd^{II}} \triangleq B^e_{-c^I} = \begin{bmatrix} \frac{5}{200} & 0 & 0 \\ 0 & \frac{35}{80} & \frac{4}{400} \end{bmatrix}, B^e_{c^I} = \begin{bmatrix} \frac{10}{31} \\ \frac{9}{31} \end{bmatrix}. \tag{114}$$

Then, via Eq. (40), we get

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B^e_{-c^I} L^{(-e-c^I)} Y^{(-e-c^I)} + B^e_{c^I} B^{c^I}_{-c^I} L^{(-e-c^I)} Y^{(-e-c^I)} + B^e_{c^I} Y^{c^I} = \begin{bmatrix} 15 \\ 48 \end{bmatrix}. \tag{115}$$

This outcome again agrees exactly with the known values of waste released into the environment. If the last term is neglected, then

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B^e_{-c^I} L^{(-e-c^I)} Y^{(-e-c^I)} + B^e_{c^I} B^{c^I}_{-c^I} L^{(-e-c^I)} Y^{(-e-c^I)} = \begin{bmatrix} 14.7 \\ 47.7 \end{bmatrix} \approx \begin{bmatrix} 15 \\ 48 \end{bmatrix}, \tag{116}$$

which is still very close to the exact above results. However, if only the first term is included, then

$$\begin{bmatrix} X^{e_1} \\ X^{e_2} \end{bmatrix} = B^e_{-c^I} L^{(-e-c^I)} Y^{(-e-c^I)} = \begin{bmatrix} 5 \\ 39 \end{bmatrix} \neq \begin{bmatrix} 15 \\ 48 \end{bmatrix}, \tag{117}$$

which is quite far from the exact result.

Per one unit product fd^I (only $\delta Y^{fd^I}_{c^I} = 1$ otherwise $\delta Y^{j \neq fd^I}_{c^I} = 0$), C^{II} ($\delta Y^{c^{II}}_{c^I} = 1$), or fd^{II} ($\delta Y^{fd^{II}}_{c^I} = 1$) needed by C^I, we have

$$\delta Y^{(-e-c^I)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{118}$$

Then, via Eq. (40), the corresponding decomposition-form wastes to environment is respectively

$$\begin{bmatrix} \delta X^{e^I} \\ \delta X^{e^{II}} \end{bmatrix} = \begin{bmatrix} \frac{537}{8680} \\ \frac{1497}{8680} \end{bmatrix}, \begin{bmatrix} \frac{50}{217} \\ \frac{1434}{1085} \end{bmatrix}, \begin{bmatrix} \frac{4}{217} \\ \frac{8353}{86800} \end{bmatrix}. \tag{119}$$

Comparing the environmental burden due to per unit of product of C^{II}, as needed by respectively sector $fd = fd^I + fd^{II}$, sector fd^I and sector c^I, we have,

$$\begin{bmatrix} \delta X^{e^I} \\ \delta X^{e^{II}} \end{bmatrix} = \begin{bmatrix} \frac{3}{16} \\ \frac{27}{32} \end{bmatrix}_{1C^{II}\to fd}, \begin{bmatrix} \frac{5}{32} \\ \frac{279}{160} \end{bmatrix}_{1C^{II}\to fd^I}, \begin{bmatrix} \frac{50}{217} \\ \frac{1434}{1085} \end{bmatrix}_{1C^{II}\to c^I}. \tag{120}$$

They are all different. This finding means that, even if the same unit of a product is needed, it matters who is buying or consuming the product.

Even if we consider percentage, e.g. fd total, fd^I and c^I all needs one percent more of product C^{II} which is respectively $(5, 2.5, 1)$ units of product C^{II}, is required and thus correspondingly the above three vectors must time these three values respectively, they are still different,

$$\begin{bmatrix} \delta X^{e^I} \\ \delta X^{e^{II}} \end{bmatrix} = \begin{bmatrix} 0.94 \\ 4.22 \end{bmatrix}_{1\%\left(C^{II}\to fd\right)}, \begin{bmatrix} 1.17 \\ 4.35 \end{bmatrix}_{1\%\left(C^{II}\to fd^I\right)}, \begin{bmatrix} 0.23 \\ 1.32 \end{bmatrix}_{1\%\left(C^{II}\to c^I\right)}. \tag{121}$$

All of this environmental burden due to one unit of product or one percent of the products of one sector that are required by other sectors might be informative for decision making in management or monitoring of the combined economic and environmental system.

Finally, let us illustrate the application of FIOA to pure economic systems of a multiregion setting.

3.5. *Pure economic multiregion input-output analysis*

This is to simply ignore the environmental sectors from Table 4 and the matrix in Eq. (89), thus, the full direct input-output coefficient matrix of the economic system only in this case is,

$$B = \begin{bmatrix} \frac{1}{31} & \frac{5}{200} & \frac{20}{80} & \frac{5}{400} \\ \frac{200}{31} & 0 & 0 & 0 \\ \frac{10}{31} & \frac{25}{200} & \frac{20}{80} & \frac{25}{400} \\ 0 & 0 & \frac{400}{80} & 0 \end{bmatrix}. \tag{122}$$

Now, let us apply LIOA (Eq. (11)) and TIOA (Eq. (33)), both of which are special cases of FIOA (Eq. (29)) to this input-output matrix.

3.5.1. *Due to final demands*

Let us first work out LIOA (Eq. (11)) for the sector of final demands $fd = fd^I + fd^{II}$, which needs

$$B^{(-fd)} = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} \\ \frac{10}{31} & \frac{20}{80} \end{bmatrix}, \tag{123}$$

thus

$$L^{(-fd)} = \begin{bmatrix} \frac{93}{89} & \frac{31}{89} \\ \frac{4}{89} & \frac{120}{89} \end{bmatrix}. \tag{124}$$

Thus, from Eq. (11), when

$$Y = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \tag{125}$$

we obtain

$$X = L^{-fd}Y = \begin{bmatrix} 31 \\ 80 \end{bmatrix}, \tag{126}$$

which agrees exactly with the total output from sectors c^I and c^{II}. In fact, the whole calculation is the same as the calculation in §3.2.1 although bearing a different meaning (the corresponding sectors in two regions instead of two sectors in the same region). Thus, we will skip the rest.

Here comes the interesting part. Let us study the economics of impacts due to fd^I only, instead of $fd = fd^I + fd^{II}$. In this case, the direct input-output matrix is

$$B^{(-fd)} = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} & \frac{5}{400} \\ \frac{10}{31} & \frac{20}{80} & \frac{25}{400} \\ 0 & \frac{400}{80} & 0 \end{bmatrix}, \tag{127}$$

thus

$$L^{(-fd)} = \begin{bmatrix} \frac{217}{160} & \frac{31}{32} & \frac{31}{400} \\ 1 & 3 & \frac{1}{5} \\ 5 & 15 & 2 \end{bmatrix}. \tag{128}$$

Thus, from Eq. (11), when

$$Y = \begin{bmatrix} 5 \\ 25 \\ 0 \end{bmatrix}, \tag{129}$$

we get

$$X = L^{-fd}Y = \begin{bmatrix} 31 \\ 80 \\ 400 \end{bmatrix}, \tag{130}$$

which again agrees exactly with the total output from sector c^I, c^{II} and fd^{II}. More importantly, the decomposition reads, when

$$\delta Y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{131}$$

we obtain

$$\delta X = \begin{bmatrix} \frac{217}{160} \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} \frac{31}{32} \\ 3 \\ 15 \end{bmatrix}, \begin{bmatrix} \frac{31}{400} \\ \frac{1}{5} \\ 2 \end{bmatrix}. \tag{132}$$

From these outcomes, we see that even fd^{II} does not directly export any workforce to region I ($x_{c^I}^{fd^{II}} = 0$, $x_{fd^I}^{fd^{II}} = 0$), per unit of product C^I consumed by df^I, in fact, includes 5 hours of labor from fd^{II}, together with $\frac{217}{160}$ units of product C^I and 1 units of product of C^{II}. Furthermore, per product of C^{II} consumed by fd^I in fact means that the whole economy must produce $\frac{31}{32}$ unit of product C^I, 3 unit of C^{II}, and 15 hours of labor from fd^{II}.

3.5.2. Due to other economic sectors

Instead of removing the fds from Table 4 and Eq. (89), let us now remove sector c^I to study its economic impacts. In this case, the direct input-output matrix is

$$B^{(-c^I)} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{25}{200} & \frac{20}{80} & \frac{25}{400} \\ 0 & \frac{400}{80} & 0 \end{bmatrix}, \tag{133}$$

thus

$$L^{(-c^I)} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{7} & \frac{16}{7} & \frac{1}{7} \\ \frac{10}{7} & \frac{80}{7} & \frac{12}{7} \end{bmatrix}. \tag{134}$$

Thus, from Eq. (33), when

$$Y = \begin{bmatrix} 200 \\ 10 \\ 0 \end{bmatrix}, \tag{135}$$

we get

$$X = L^{(-c^I)}Y = \begin{bmatrix} 200 \\ 80 \\ 400 \end{bmatrix}, \tag{136}$$

which again agrees exactly with the total output from sector c^{II}, fd^I and fd^{II}. More importantly, the decomposition reads, when

$$\delta Y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{137}$$

we obtain

$$\delta X = \begin{bmatrix} 1 \\ \frac{2}{7} \\ \frac{10}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{16}{7} \\ \frac{80}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{7} \\ \frac{12}{7} \end{bmatrix}. \tag{138}$$

From these calculations, we see that even fd^{II} does not directly export any workforce to region I ($x_{c^I}^{fd^{II}} = 0$, $x_{fd^I}^{fd^{II}} = 0$), per unit of labor fd^I needed for production of C^I, in fact, includes 1 hours of labor from fd^I, $\frac{2}{7}$ units of product C^{II} and $\frac{10}{7}$ hours of labor from fd^{II}. This is due to the fact that sector c^I needs input from fd^I, then fd^I requires input from c^{II}, and in turn fd^{II} supplies a workforce for c^{II}. Thus all orders of effects are taken into account by the FIOA.

Again we see that economic impacts due to per unit of C^{II}, when required by either fd^I or c^I, are different,

$$\begin{bmatrix} \delta X^{c^{II}} \\ \delta X^{fd^{II}} \end{bmatrix}_{c^{II} \to fd^I} = \begin{bmatrix} \cancel{\frac{31}{32}} \\ 3 \\ 15 \end{bmatrix} \neq \begin{bmatrix} \delta X^{c^{II}} \\ \delta X^{fd^{II}} \end{bmatrix}_{c^{II} \to c^I} = \begin{bmatrix} \cancel{0} \\ \frac{16}{7} \\ \frac{80}{7} \end{bmatrix} \tag{139}$$

even after excluding the clearly different entries: the first entry in the first vector corresponding to δX^{c^I} and the first entry in the second vector corresponding to δX^{fd^I}, are totally not comparable and thus they are removed. This means that as we observed earlier the impact depends on who is using it, even when the same amount of the same product is needed by different sectors.

3.6. Re-conducting multiregion input-output analysis via the matrix-notation formulae

To illustrate the usefulness of the matrix-notation formulae, Eq. (47) and Eq. (48), we now re-calculate the economic and environmental impacts using the matrix-notation formulae, taking again respectively c^I and fd^I as the external sector.

The full direct input-output coefficient matrix B is given in Eq. (89), from which we obtain emissions matrix,

$$B^e_{\text{econ}} = \begin{bmatrix} \frac{10}{31} & \frac{5}{200} & 0 & 0 \\ \frac{9}{31} & 0 & \frac{35}{80} & \frac{4}{400} \end{bmatrix}, \tag{140}$$

and input-output coefficients on flows from c^I to other sectors,

$$B^{c^I}_{-c^I} = \begin{bmatrix} \frac{5}{200} & \frac{20}{80} & \frac{5}{400} \end{bmatrix}, B^{c^I}_{c^I} = \begin{bmatrix} \frac{1}{31} \end{bmatrix}. \tag{141}$$

We then need

$$B^{(-e-c^I)} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{25}{200} & \frac{20}{80} & \frac{25}{400} \\ 0 & \frac{400}{80} & 0 \end{bmatrix}, \tag{142}$$

and

$$L^{(-e-c^I)} = \frac{1}{1 - B^{(-e-c^I)}} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{7} & \frac{16}{7} & \frac{1}{7} \\ \frac{10}{7} & \frac{80}{7} & \frac{12}{7} \end{bmatrix}, \tag{143}$$

which are provided previously. Based on Eq. (47), we construct the complete input-output matrix $\bar{\mathcal{L}}^{-e-fd^I}$,

$$\bar{\mathcal{L}}^{-e-fd^I} = \begin{bmatrix} 1 & \frac{4}{35} & \frac{5}{7} & \frac{2}{35} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{16}{7} & \frac{1}{7} \\ 0 & \frac{10}{7} & \frac{80}{7} & \frac{12}{7} \end{bmatrix}, \tag{144}$$

which captures all of the information in a decomposition form regarding the pure economic impacts of all input-output flow into sector c^I. Thus, according to Eq. (48), the matrix, revealing all of the information in a decomposition form regarding the environmental impacts of all input-output flow into sector c^I, is

$$\mathcal{L}^{e-c^I} = B^e_{\text{econ}} \bar{\mathcal{L}}^{-e-c^I} = \begin{bmatrix} \frac{10}{31} & \frac{537}{8680} & \frac{50}{217} & \frac{4}{217} \\ \frac{9}{31} & \frac{1497}{8680} & \frac{1434}{1085} & \frac{8353}{86800} \end{bmatrix}. \tag{145}$$

These $\bar{\mathcal{L}}^{-e-c^I}$ and $\mathcal{L}^{e-c^I}$ agree exactly with the corresponding results in the previous subsection.

When fd^I is taken to be the external sector, a fully parallel calculation needs

$$B^{fd^I}_{-fd} = \begin{bmatrix} \frac{200}{31} & 0 & 0 \end{bmatrix}, B^{fd^I}_{fd} = \begin{bmatrix} 0 \end{bmatrix}. \tag{146}$$

and the previously known input-output matrix

$$B^{(-e-fd^I)} = \begin{bmatrix} \frac{1}{31} & \frac{20}{80} & \frac{5}{400} \\ \frac{10}{31} & \frac{20}{80} & \frac{25}{400} \\ 0 & \frac{400}{80} & 0 \end{bmatrix}. \tag{147}$$

and

$$L^{(-e-fd^I)} = \frac{1}{1 - B^{(-e-fd^I)}} = \begin{bmatrix} \frac{217}{160} & \frac{31}{32} & \frac{31}{400} \\ 1 & 3 & \frac{1}{5} \\ 5 & 15 & 2 \end{bmatrix}. \tag{148}$$

Based on Eq. (47), we construct the complete input-output matrix $\bar{\mathcal{L}}^{-e-c^I}$,

$$\bar{\mathcal{L}}^{-e-c^I} = \begin{bmatrix} \frac{217}{160} & 0 & \frac{31}{32} & \frac{31}{400} \\ \frac{35}{4} & 1 & \frac{25}{4} & \frac{1}{2} \\ 1 & 0 & 3 & \frac{1}{5} \\ 5 & 0 & 15 & 2 \end{bmatrix}, \tag{149}$$

which captures all of information in a decomposition form regarding the pure economic impacts of all input-output flow into sector c^I. Thus, according to Eq. (48), the matrix, revealing all of the information in a decomposition form regarding environmental impacts of all input-output flow into sector c^I, is

$$\mathcal{L}^{e-fd^I} = B^e_{\text{econ}} \bar{\mathcal{L}}^{-e-fd^I} = \begin{bmatrix} \frac{21}{32} & \frac{1}{40} & \frac{15}{32} & \frac{3}{80} \\ \frac{141}{160} & 0 & \frac{279}{160} & \frac{13}{100} \end{bmatrix}. \tag{150}$$

These $\bar{\mathcal{L}}^{-e-fd^I}$ and $\mathcal{L}^{e-fd^I}$ agree exactly with the corresponding results in the previous subsection. We can see clearly that performing calculations in this way for arbitrary l, no matter it is an economic sector (c) or a sector of final demanders (fd), follows exactly the same steps and uses exactly the same formulae. Thus, we believe that this matrix-notation formulae is more practical.

4. Conclusion and discussion

In all of the above examples and in the presentation of the methods, we start from a full input-output table x, where the input-output flow between each pair of sectors x_j^i are known, including those that go into or come from environmental sectors. In reality, it might not be the case. Sometimes, even the value-added flow, x_j^{fd}, is not readily known to researchers. In that case, often a balance check can be done so that in a monetary unit, $X^j = X^j$, where $X^j = \sum_i x_i^j + x_{fd}^j$ and $X_j = \sum_i x_j^i = \sum_i x_j^i + x_j^{fd}$. Thus, $x_j^{fd} = \sum_i x_i^j - \sum_i x_j^i + x_{fd}^j$. Furthermore, often the flow between environmental sectors and other sectors is not known to researchers. However in such cases, the emissions vector, $f_j^{e_i} = B_j^{e_i}$, which is the amount of waste or resources required to produce one unit of product j, is expected to be known. Therefore, the full direct input-output coefficient matrix B can then often be constructed. In this sense, we assume that, in some way, the full matrix B is always accessible to researchers.

Once we have this full matrix B, if we are interested in the economic impacts of an input-output flow from an economic sector to the sector of final demanders (fd), then LIOA, i.e. Eq. (11) can be used, where fd is treated as the external sector. If the economic impacts of an input-output flow from an economic sector to another specific economic sector (c_1), then i.e. Eq. (33) can be used, where sector c_1 is regarded as the external sector.

If we are interested in environmental impacts of an input-output flow from an economic sector to the sector of final demanders (fd), then Eq. (18) can be used, where fd is treated as the external sector. Note that, even in this case, Eq. (18) leads to more accurate results than the widely used formulae for environmental input-output analysis, Eq. (2)/Eq. (19), at least in our toy model. It is also beyond Eq. (22), the formula suggested recently by Hubacek et al. (Hubacek et al., 2017) since Eq. (18) is in a fully decomposition form, while Eq. (22) is partially in a form of total amount. If environmental impacts of an input-output flow from an economic sector to another specific economic sector (c_1), then i.e. Eq. (40) can be used, where sector c_1 is regarded as the external sector.

In principle, input-output analysis should be applicable to studies of the economic or environmental impact of an input-output flow from an economic sector to an environmental sector. However, in the current form, this can not be undertaken since the input-output flows from economic sectors to environmental sectors ($x_{e_j}^{c_i} = 0$) are assumed to be zero such that $\frac{1}{1-B^{(-e)}}$ diverges. In the future, if necessary, environmental sectors can be treated like full input-output sectors, which rely on economic sector to sustain such that some $x_{e_j}^{c_i} \neq 0$, then, the above question can also be answered.

The method is readily applicable to multiregion input-output analysis, in which the economic or environmental impacts of an input-output flow from an arbitrary economic sector to fds in each region, such as fd^{I} and fd^{II}, or the total fd in both region can be investigated. The same holds for the impact of an input-output flow between any two economic sectors, such as using sector c^{I}_{j}, an economic sector j in region I, as the external sector.

Again, recalling the fact that all of the formulae come from FIOA, Eq. (29), we hence present a unified framework of input-output analysis, as evidently shown by the unified formulae Eq. (47) and Eq. (48).

We limited our analysis in this work to demands-driven analysis and input-output tables in physical units. A similar supply-driven analysis can be done in parallel, where matrix B is replaced by a matrix F, where $F^{i}_{j} = \frac{x^{i}_{j}}{X_i}$, and $B^{(-l)}$ is replaced by $F^{(-l)}$. This F-based analysis is suitable for studies of the economic/environmental impacts of a supply from the targeted sector l to all other sectors. In this case, l can even be environmental sectors to investigate economic/environmental impacts of a supply from the environmental sector l to all other sectors. The formulae will be very much the same, however, their applicability will be the topic of future studies. Applications of the method to input-output tables in monetary units is straightforward.

Furthermore, the method is readily applicable to systems that beyond economic-environmental system such as citation networks of scientific fields, papers, patents, or papers and patents (Shen et al., 2016), as long as a full input-output matrix x can be constructed. For example, in the case of citation networks of scientific fields, x^{i}_{j} means the number of citations that papers in fields i are cited by papers in fields j (Shen et al., 2016).

The main contribution is summarized in Fig. 2. It also serves as a quick reference to choose the proper formula to use for each purpose. As shown previously via toy-model data, for the analysis of the environmental impacts, our decomposition-form Eq. (18) is more accurate than the commonly used Eq. (19), and it agrees with and exceeds the aggregated-level Eq. (22). Often, the decomposition form is preferred more since the results from the aggregated-form formula can always be derived from results from the decomposition-form formula, but not the other way around. Furthermore, for analyses of both the economic impact and the environmental impacts, our decomposition-form Eq. (33) and Eq. (40) are capable of addressing an arbitrary economic sector in any region or the combined region as the external sector, in addition to the sector of final demands. These are

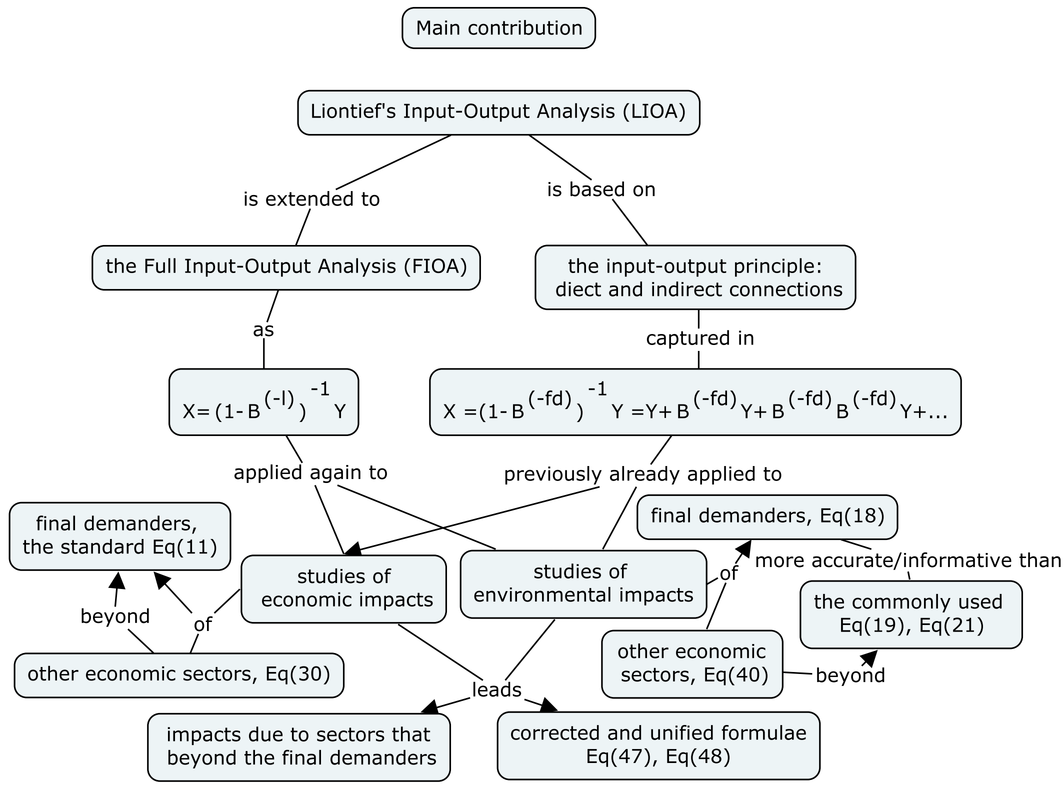

Figure 2: Main contribution of the current work, also serving as a graphic abstract. Note that the unified formulae Eq. (47) and Eq. (48) are based on the unified idea – the input-output principle.

the major contribution of the current work.

To ease the task of choosing which formulae to use for specific purposes, we also present the core formula in a matrix-notation form, which appears to be more abstract, but allows us to follow exactly the same steps and the same formulae when choosing an arbitrary sector as the external sector and for analysis of both economic and environmental impacts. Additionally, since it is in a matrix form, the formula is fully in a decomposition form. Furthermore, such a set of unified formulae shows that indeed all the above input-output analysis is in fact variation of a common principle.

5. Acknowledgements

A program that accepts a full input-output table (x) or a full input-output matrix (B), and accepts the sector that should be used as the external sector (l) as in-

put, and calculates economic and environmental impacts as output, can be found at

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